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A Polynomial of Primes

634. [September, 1966] *Proposed by R. S. Luthar and Stephen Wurzel, Colby College, Maine.*

If p is a prime, such that

$$p^2 \not\equiv p \pmod 3$$

show that

$$p^{2n-1} + p^{2n-3} + \cdots + p + n \equiv 0 \pmod 3.$$

Solution by Stanley Rabinowitz, Far Rockaway, New York.

The conclusion is true if p is any number relatively prime to 3. If $p^2 - p \not\equiv 0 \pmod 3$ and $(p, 3) = 1$, then $p \not\equiv 0 \pmod 3$, and so $p - 1 \not\equiv 0 \pmod 3$. The only other case is that $p \equiv -1 \pmod 3$. Since by Fermat's Theorem, $p^2 \equiv 1 \pmod 3$, we also have $p^3 \equiv -1$, $p^5 \equiv -1$, $p^7 \equiv -1, \cdots, \pmod 3$. Hence $p^{2n-1} + p^{2n-3} + \cdots p^3 + p \equiv (-1) + (-1) + \cdots + (-1) + (-1) \equiv -n \pmod 3$ and the result follows.